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Solution of the Schrödinger equation of the complex manifold CP^n

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Abstract

Passing from the $CP^n = SU(n+1)/U(n)$ Lagrangian in Kähler form, to the Hamiltonian in terms of polar coordinates, this paper describes the solutions of the CP^n Schrödinger equation, whose energy eigenspaces carry the irreducible representations $(\lambda, 0^{n-2}, \lambda)$, $\lambda = 0, 1, 2, \ldots$, of $SU(n+1)$ in highest weight notation. A full account is given of the $U(n)$ structure of these eigenspaces, and of how this emerges within our solution of the Schrödinger equation by separation of variables. Explicit solutions of the radial equation, which give rise to the derivation of spectrum and energy eigenspace details, are presented for $\lambda = 1, 2$, and related to Jacobi polynomials.

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1. Introduction

In this paper, we undertake the solution of the Schrödinger equation

$$H_q \Psi = E\Psi \tag{1}$$

of the complex manifold CP^n. This requires the extension from $n = 2$ and $CP^2 = SU(3)/U(2)$ of the treatment given in [1] to CP^n for any integer n. Our previous paper gave a brief indication of our motivation, and of the status of CP^n models in general in various areas of theoretical physics, which is not repeated here.

As stated in [1] the spectrum (see (4)) of the $CP^n = SU(n+1)/U(n)$ Schrödinger equation is known, having been given in [2]. The latter source also states which irreducible representations (irreps) of $SU(n+1)$ are carried by the energy eigenspaces of (1). These, in highest weight notation, see e.g., [3, 4], are the irreps

$$(\lambda, 0^{n-2}, \lambda) \qquad \lambda \in \{0, 1, 2, \ldots\} \tag{2}$$

here designated the class one irreps of $SU(n+1)$, in accordance with the definition found in [5], for which some background discussion is provided in [1].

For $n = 2$, the solution of (1) is given in [1] in sufficient detail to make clear exactly how those irreps of $U(2)$ that occur upon the restriction of the $SU(3)$ irrep (λ, λ) to $U(2)$ may be seen to exhaust the energy eigenspace of energy

$$E = 2\lambda(\lambda + 2) \tag{3}$$

for $\lambda = 0, 1, 2, \ldots$. In this paper, our aim is to extend the treatment from $n = 2$ and CP^2, to the case of general n, and give the corresponding analysis of the energy eigenspaces of (1), which have energies

$$E = 2\lambda(\lambda + n) \tag{4}$$

for $\lambda = 0, 1, 2, \ldots$.

For the purpose stated, it is necessary to assemble some group theoretic information, not all of which are immediately available in the literature. First, we prove by group theoretic means that the class one irreps of $SU(n+1)$ are given by (2). Second, we determine the decompositions into irreps of $U(n)$ that arise by their restriction to this subgroup of $SU(n+1)$. Third, we collect all the results needed for later use regarding the dimensions of irreps of unitary groups and the eigenvalues of their quadratic Casimir operators.

That we might meet some obstacles in the way of direct generalization from $n = 2$ to general n of the methods of [1] can readily be appreciated. The irreps of $SU(2)$ that occur in the restriction from $SU(3)$ to $U(2)$ of any class one (or indeed any) irrep of $SU(3)$ depend only on one variable, the total isospin I. But, as our group theoretic studies show, the irreps of $SU(3)$ that arise by restriction from $SU(n+1)$ to $U(n)$ of the class one irreps (2) of $SU(n+1)$ require, for all $n \geqslant 3$, two variables. The implications of this fact manifest themselves in the separation of variables for (1) for CP^n at the point at which the key equation (35) needs to be employed. As can be seen, this equation involves a non-trivial complication that is absent only in the degenerate $n = 2$ case. The resolution (35) of it is, further, essential for the completion of the separation of variables for $n \geqslant 3$.

Our approach to solving the Schrödinger equation (1) for CP^n follows much the same lines as did [1] for the case $n = 2$. We use well-known formulae for the Lagrangian L, Hamiltonian H and the $SU(n+1)$ transformation properties of the n complex dynamical variables K_p, $p \in \{1, 2, \ldots, n\}$ employed in standard descriptions of CP^n models [6, 7]. We define the variables K_i in terms of polar coordinates, write L in terms of them, thereby exposing the CP^n metric tensor g_{ab}, where a, b refer to the polar coordinates. This enables us to calculate explicitly the quantum Hamiltonian of the CP^n Schrödinger equation

$$H_q = -\tfrac{1}{2} g^{-1/2} \partial_a g^{1/2} g^{ab} \partial_b. \tag{5}$$

We wish to solve this equation by separation of variables with the aim of reaching a radial equation, on the basis of which we expect to prove that the $U(n)$ subspaces of its energy eigenspaces are indeed the ones expected from the decompositions previously deduced for the $SU(n+1)$ irreps (2). To achieve this we must calculate explicitly the quadratic Casimir of $SU(n)$ in terms of our polar coordinates, which of course is expected not to involve the radial coordinate r. This is necessary to enable us to see exactly its place (see (35)) in our separation of variables procedure, and to allow the correct form of the final radial equation to emerge. Even in the first non-trivial case beyond $n = 2$, $CP^3 = SU(4)/U(3)$, where we use polars defined by

$$\begin{aligned} K_1 &= r \cos\sigma \cos\tfrac{1}{2}\theta \, \mathrm{e}^{\mathrm{i}\alpha} \\ K_2 &= r \cos\sigma \sin\tfrac{1}{2}\theta \, \mathrm{e}^{\mathrm{i}\beta} \\ K_3 &= r \sin\sigma \, \mathrm{e}^{\mathrm{i}\gamma} \end{aligned} \tag{6}$$

so that $r = \sqrt{\sum_{p=1}^{3}(\bar{K}_p K_p)}$, it is tedious to evaluate the $SU(3)$ generators, and hence the quadratic Casimir $\mathcal{C}^{(2)}$, in terms of the polar angles $\sigma, \theta, \alpha, \beta, \gamma$ and their derivatives. We confine ourselves to supplying full details in this case, but eventually we reach the desired radial equation for the CP^n Schrödinger equation. We verify that it has the expected spectrum and that its energy eigenspaces have the descriptions in terms of irreps of $U(n)$ expected on the basis of our group theoretic analysis, and give explicit radial eigenfunctions for all the states of the irreps (2) of $SU(n+1)$ for $\lambda = 1$ and 2 in terms of Jacobi polynomials. The results obtained previously [1] for $n = 2$ conform to the same general patterns.

The content of the remaining sections of this paper is as follows. In section 2, we discuss irreps of $SU(n+1)$, especially those of (2), and their $U(n)$ decompositions. Section 3 assembles the data required for later use about the dimension and eigenvalues $c_{2,(n+1)}$ and $c_{2,n}$ of such irreps. Section 4 describes results for CP^n models needed to obtain their Schrödinger equations, and to embark on solution by separation of variables, giving full detail at some points only for $n = 3$. Section 5 then discusses spectrum, energy eigenspace structure and explicit radial wavefunctions for CP^n in general.

2. Class one irreps of $SU(n+1)$

We know on general grounds, summarized in section 8 of [1] that the eigenspaces of (1) for $CP^n = SU(n+1)/U(n)$ are the carrier spaces of the class one irreps of $SU(n+1)$ relative to its massive $U(n)$ subgroup. Here we note that a class one irrep of $SU(n+1)$ is one for which the restriction to $U(n)$ contains the identity irrep of $U(n)$. Each class one irrep of $SU(n+1)$ contains this identity once and once only, which is why the subgroup $U(n)$ is said to be a massive subgroup of $SU(n+1)$.

Accordingly, we must identify the class one irreps of $SU(n+1)$, and assemble the information regarding them that is required for our use below.

It is convenient to make our deductions in terms of the Young tableaux description $\{l_1, l_2, \ldots, l_n\}$, where the l_i are integers such that $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_n \geqslant 0$, but to state our results for actual use in terms of highest weight notation $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where the λ_i are integers ($\geqslant 0$). We use braces in the former and round brackets in the latter context. The connection between them is

$$\lambda_i = l_i - l_{i+1} \qquad i = 1, 2, \ldots, (n-1) \qquad \lambda_n = l_n. \tag{7}$$

We consider first the decomposition of an $SU(n+1)$ irrep into irreps of $U(n)$ for the cases $n \geqslant 3$. We have excluded $n = 2$ at this point only because of a slight degeneracy of the general notation used when we set $n = 2$; the actual pattern of the results for $n = 2$ conforms in its essentials to the general n treatment. The irrep

$$\{l_1, l_2, \ldots, l_n\} \tag{8}$$

of $SU(n+1)$ contains, see e.g., [8], the irrep

$$\{m_1 - m_n, m_2 - m_n, \ldots, m_{n-1} - m_n\} \otimes y^{(n)} \tag{9}$$

of $SU(n) \otimes U(1)$, where

$$y^{(n)} = m - \frac{n}{n+1} l \qquad l = \sum_{k=1}^{n} l_k \qquad m = \sum_{k=1}^{n} m_k \tag{10}$$

exactly once for each distinct ordered set of integers

$$m_1, m_2, \ldots, m_n \tag{11}$$

allowed by the inequalities

$$l_1 \geqslant m_1 \geqslant l_2 \cdots l_n \geqslant m_n \geqslant 0. \tag{12}$$

This decomposition contains the trivial irrep of $SU(n)$ iff

$$m_1 = l_2 = m_2 = \cdots = m_{n-1} = l_n = m_n \tag{13}$$

and this is associated with $y^{(n)} = 0$, so that the irrep (8) is of class one, iff

$$l_1 = 2l_2. \tag{14}$$

It follows immediately that the set of class one irreps of $SU(n+1)$ is given by

$$(\lambda, 0^{n-2}, \lambda) \qquad \lambda = l_2 \in \{0, 1, 2, \ldots\} \tag{15}$$

where the notation indicates that there are $n-2$ highest weight components equal to zero.

For class one irreps, in Young tableaux notation $\{2\lambda, \lambda^{n-1}\}$, it follows above statements that their $U(n)$ decompositions are governed by the inequalities

$$2\lambda \geqslant m_1 \geqslant \lambda = m_2 = \cdots = m_{n-1} = l_n = m_n \geqslant 0. \tag{16}$$

Therefore the $SU(n)$ irreps involved are given by

$$\{m_1 - m_n, \lambda - m_n, \ldots, \lambda - m_n\} = (m_1 - \lambda, 0^{n-3}, \lambda - m_n) \tag{17}$$

in association with the $y^{(n)}$ value

$$y^{(n)} = m_1 + m_n - 2\lambda. \tag{18}$$

There are thus in all $(\lambda+1)^2$ such $U(n)$ irreps in the decomposition, with $y^{(n)}$ values such that

$$\lambda \geqslant y^{(n)} \geqslant -\lambda. \tag{19}$$

Having reached the results (17) and (18), it becomes clear that we can recast them in a more convenient and elegant form by defining integers $f \geqslant 0$ and $g \geqslant 0$ via

$$f = m_1 - \lambda \qquad g = \lambda - m_n. \tag{20}$$

Then, for each pair (f, g) of ordered integers $f, g \geqslant 0$, the decomposition of the irrep (15) of $SU(n+1)$ contains exactly once each irrep of $U(n)$ given by

$$(f, 0^{n-3}, g) \qquad y^{(n)} = f - g. \tag{21}$$

It is just a minor matter of adjusting our notation in the case $n = 2$ to see that the corresponding results conform to the pattern just found for general n. The class one irreps (λ, λ) of $SU(3)$ for $\lambda = 0, 1, 2, \ldots$, decompose into irreps of $U(2)$, where the $SU(2)$ factor refers to isospin I, and the $U(1)$ factor to the subgroup with hypercharge generator $Y = Y^{(2)}$. For each distinct pair of integers (f, g), $f, g \geqslant 0$, we find in the decomposition exactly one (I, Y) pair given by

$$I = \tfrac{1}{2}(f+g) \quad Y = f - g. \tag{22}$$

Put otherwise, for each integral value of $I + \frac{1}{2}|Y|$ such that $0 \leqslant I + \frac{1}{2}|Y| \leqslant \lambda$ we get one $(I, 0)$ pair if $Y = 0$, and one of each of the pairs $(I, \pm Y)$ if $Y \neq 0$.

To clarify the scale of the $y^{(n)}$ eigenvalues, we note that we use $SU(n+1)$ generators X_i, $1 \leqslant i \leqslant N$, $N = (n+1)^2 - 1 = n(n+2)$ such that

$$[X_i, X_j] = i f_{ijk} X_k \tag{23}$$

and represented in the defining irrep def via $X_i \mapsto \frac{1}{2}\lambda_i$, where we use a standard set [9] of lambda matrices normalized according to $\mathrm{Tr}\,\lambda_i\lambda_j = 2\delta_{ij}$. Our two uses of subscripted lambdas should be easily distinguishable from their contexts.

The generators $Y^{(n)}$ of the $U(1)$ subgroup of $U(n) \subset SU(n+1)$, whose eigenvalues $y^{(n)}$ feature in the results (10) and so on are related to the 'last' Gell-Mann matrix λ_N of the family

$$\lambda_N = c_n \begin{pmatrix} I_n & 0 \\ 0 & -n \end{pmatrix} \qquad c_n = \sqrt{\frac{2}{n(n+1)}} \tag{24}$$

by means of

$$Y^{(n)} \mapsto \sqrt{\frac{2n}{n+1}}\lambda_N = \frac{1}{n+1}\begin{pmatrix} I_n & 0 \\ 0 & -n \end{pmatrix}. \tag{25}$$

For the case of $SU(3)$ and $Y^{(2)} = Y$, hypercharge, this reads as

$$Y = \frac{1}{3}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} = \frac{1}{\sqrt{3}}\lambda_8. \tag{26}$$

We note also that the quadratic Casimir operator $\mathcal{C}^{(2)}$ of $SU(n+1)$ is defined by

$$\mathcal{C}^{(2)} = X_i X_i. \tag{27}$$

The normalization of its eigenvalues employed in this paper can be seen by reference either to the defining irreps for which well-known properties of Gell-Mann lambda matrices [9] give

$$c_{2,(n+1)}(def) = \tfrac{4}{3} \tag{28}$$

or else to the adjoint irrep ad, $X_i \mapsto F_i$, $(F_i)_{jk} = -\mathrm{i} f_{ijk}$ where we have [9]

$$(X_i X_i)_{kl} = f_{ijk} f_{ijl} = (n+1)\delta_{kl} \tag{29}$$

so that

$$c_{2,(n+1)}(ad) = n + 1. \tag{30}$$

3. Dimension formulae and eigenvalues of Casimir Operators

From the Weyl formula, see e.g., equation (5.5) of [4], it is easy to learn that the class one irrep $R_\lambda = (\lambda, 0^{n-2}, \lambda)$ of $SU(n+1)$ has dimension

$$\dim R_\lambda = \frac{2\lambda + n}{n}\binom{\lambda + n - 1}{\lambda}^2 \tag{31}$$

so that, for $\lambda = 1$ and the irrep ad, we have

$$\dim ad = n(n+2) = (n+1)^2 - 1. \tag{32}$$

Next we turn to the eigenvalues $c_{2,(n+1)}$ and $c_{2,n}$ of the quadratic Casimir operators $\mathcal{C}^{(2)}$ of $SU(n+1)$ and $SU(n)$. The normalization of such results is seen by reference to the irreps ad and (29). Accordingly, we use equation (4.11) and the first entry of table 7 of [4], adjusted to our normalizations. In fact it is sufficient for the needs of this paper, to quote, for $SU(n+1)$, the following, which contains a scalar product in its first two lines,

$$\begin{aligned} c_{2,(n+1)}(f, 0^{n-2}, g) &= \frac{1}{2(n+1)}(fn + g, f(n-1) + 2g, \ldots, 2f + (n-1)g, nf + g) \\ &\quad (f+2, 2, \ldots, 2, g+2) \\ &= \frac{1}{2(n+1)}[nf^2 + 2fg + ng^2 + n(n+1)(f+g)]. \end{aligned} \tag{33}$$

We need this first to obtain

$$c_{2,(n+1)}(R_\lambda) = \lambda(n + \lambda) \tag{34}$$

which reduces to $(n+1)$ for $\lambda = 1$ and $R_1 = ad$, agreeing with (30).

We need it also to establish a result crucial for introducing $U(n)$ labels correctly into our final CP^n radial equation. This result is

$$4c_{2,n}(f, 0^{n-3}, g) - \left(1 - \frac{2}{n}\right)(f - g)^2 = h(h + 2n - 2) \qquad h = f + g \tag{35}$$

and is important because the right-hand side depends on a single variable h, i.e., it depends on the pair (f, g) only through the sum $h = f + g$.

The notation at this point, as previously, needs to be refined to accommodate the case of $n = 2$. It also implies that the treatment to follow, of the separation of variables for (1) in the general n case for $n > 2$, involves a complication absent in the degenerate case of $n = 2$. For $SU(3)$, and its class one irreps (λ, λ), we only need to know that the Casimir $\mathbf{I}^2$ of its $SU(2)$ isospin subgroup has eigenvalues $I(I + 1)$, $2I (\equiv h) = f + g$, as (35) for $n = 2$ makes clear.

4. Review of CP^n model

Following well-known lines dating back to the 1970s (see [6, 7, 10]), we set out from an $(n + 1)$-component column vector $Z(K)$ dependent on the complex quantities K_p, $p = 1, 2, \ldots, n$, and given by

$$Z^T = (LK_i, L) \qquad L = L(K) = (1 + X)^{-1/2} \qquad X = \sum_{p=1}^{n} \bar{K}_p K_p \tag{36}$$

so that $Z^T Z = 1$. The quantity $Z(K)$ transforms under $U \in SU(n)$ according to the law

$$Z(K) \mapsto Z(K') \qquad \text{where} \quad UZ(K) = Z(K')V(U, K). \tag{37}$$

Here $V = V(U, K)$ serves to ensure that $Z(K')_{n+1} = L(K')$ can be chosen to be real.

It follows (37) that the transformations of $U(n)$ involving $SU(n)$ and $U(1)$ are realized linearly while those whose generators lie outside the Lie algebra $su(n) + u(1)$ are realized nonlinearly, taking the infinitesimal form [6] on the variables K_p

$$\delta K = \tfrac{1}{2}\mathrm{i}[\epsilon - K(\bar{\epsilon} \cdot K)]. \tag{38}$$

The route from (37) to the CP^n Lagrangian is too well known to need review. It yields the result

$$L = g_{p\bar{q}} \dot{K}_p \dot{\bar{K}}_q \tag{39}$$

where the CP^n metric tensor is

$$g_{p\bar{q}} = (1 + X)^{-1}\delta_{pq} - (1 + X)^{-2}\bar{K}_p K_q \tag{40}$$

in its well-known Kähler form [11]. Proceeding classically at first, we define the canonical momenta Π_p, $\bar{\Pi}_p$, and obtain the Hamiltonian

$$H = g^{p\bar{q}}\Pi_p\bar{\Pi}_q = (1 + X)[\bar{\Pi} \cdot \Pi + (\Pi \cdot K)(\bar{\Pi} \cdot \bar{K})]. \tag{41}$$

Next we consider the consequences of Noether's theorem. Using the description of $SU(n + 1)$ transformations that follows (37), we can evaluate their generators in terms of canonical coordinates, check Poisson brackets and show that to within an overall multiplicative constant the classical Hamiltonian coincides with the expression in terms of canonical variables for the quadratic Casimir operator of $SU(n + 1)$. For our present purpose of treating the Schrödinger of the CP^n models, we need only expressions for the linearly realized generators, associated with the $U(n)$ subgroup, and the quadratic Casimir operator of $SU(n)$. But we need them first in terms of canonical variables and then, in practice, in terms of polar coordinates and derivatives with respect to them.

To avoid, perhaps only minimize, the inclusion of tedious detail, we refer explicitly here only to the case of $CP^3 = SU(4)/U(3)$. This is sufficient to observe the pattern and attend to the complication that exist for CP^n beyond $n = 2$. We employ notation familiar in the particle physics applications of $SU(3)$. Thus we label $SU(3)$ generators X_i which obey (23)

$$\begin{aligned} I_\pm &= X_1 \pm \mathrm{i}X_2 & I_3 &= X_3 \\ Y &= \frac{\sqrt{3}}{2}X_8 & Y &\equiv Y^{(2)} \\ V_\pm &= X_4 \pm \mathrm{i}X_5 & U_\pm &= X_6 \pm \mathrm{i}X_7. \end{aligned} \tag{42}$$

In the context of $SU(4)$, we have the independent complex variables K_p, $p = 1, 2, 3$, and we express them in terms of polar coordinates via

$$K_1 = r\cos\sigma\cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\alpha} \qquad K_2 = r\cos\sigma\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\beta} \qquad K_3 = r\sin\sigma\mathrm{e}^{\mathrm{i}\gamma}. \tag{43}$$

The detail related to the embedding of the isospin $SU(2)$ subgroup of $SU(3)$ could have been put more precisely into standard form by means of the definitions $\alpha = (\phi+\psi)/2$, $\beta = (\phi-\psi)/2$. Also the generalization from $SU(4)$ to $SU(n)$ should be evident from (43) because of the resemblance of the σ, θ dependence to spherical polar coordinates.

The end results of a body of detailed calculations can be presented now

$$\begin{aligned} &2\mathrm{i}I_3 = -\partial_\alpha + \partial_\beta \qquad 3\mathrm{i}Y = -\partial_\alpha - \partial_\beta + 2\partial_\gamma \\ &2\mathrm{i}I_+ = \mathrm{e}^{-\mathrm{i}(\alpha-\beta)}\left[2\mathrm{i}\partial_\theta - \tan\frac{\theta}{2}\partial_\alpha - \cot\frac{\theta}{2}\partial_\beta\right] \\ &2\mathrm{i}V_+ = \mathrm{e}^{-\mathrm{i}(\alpha-\gamma)}\left[\mathrm{i}\cos\frac{\theta}{2}\partial_\sigma + 2\mathrm{i}\tan\sigma\sin\frac{\theta}{2}\partial_\theta - \tan\sigma\sec\frac{\theta}{2}\partial_\alpha - \cot\sigma\cos\frac{\theta}{2}\partial_\gamma\right] \\ &2\mathrm{i}U_+ = \mathrm{e}^{-\mathrm{i}(\beta-\gamma)}\left[\mathrm{i}\sin\frac{\theta}{2}\partial_\sigma - 2\mathrm{i}\tan\sigma\cos\frac{\theta}{2}\partial_\theta - \tan\sigma\operatorname{cosec}\frac{\theta}{2}\partial_\beta - \cot\sigma\sin\frac{\theta}{2}\partial_\gamma\right]. \end{aligned} \tag{44}$$

Also for the $U(1)$ generator $Z = Y^{(3)}$ for CP^3, we have

$$\mathrm{i}Z = -(\partial_\alpha + \partial_\beta + \partial_\gamma). \tag{45}$$

The results (44) enable us to calculate the quadratic Casimir operator $\mathcal{C}^{(2)}$ of the linearly realized $SU(3)$ subgroup of $SU(4)$. We find

$$\begin{aligned} -4\mathcal{C}^{(2)} &= -4\left[\mathbf{I}^2 + \frac{3}{4}Y^2 + \frac{1}{2}(U_+U_- + U_-U_+ + V_+V_- + V_-V_+)\right] \\ &= \partial_\sigma^2 + (\cot\sigma - 3\tan\sigma)\partial_\sigma + 4\sec^2\sigma\left(\partial_\theta^2 + \cot\theta\,\partial_\theta\right) \\ &\quad + \sec^2\sigma\left(\sec^2\frac{\theta}{2}\partial_\alpha^2 + \operatorname{cosec}^2\frac{\theta}{2}\partial_\beta^2\right) + \operatorname{cosec}^2\chi\,\partial_\gamma^2 + \frac{1}{3}(\partial_\alpha + \partial_\beta + \partial_\gamma)^2. \end{aligned} \tag{46}$$

We turn next to the evaluation of the Lagrangian L of (39) for CP^n in terms of the polar coordinates (43), in order to identify the components of the metric tensor g_{ab}, $a = r, \sigma, \theta, \alpha, \beta, \gamma$, and pass thence to the quantum Hamiltonian

$$H_q = -\tfrac{1}{2}g^{-1/2}\partial_a g^{1/2}g^{ab}\partial_b. \tag{47}$$

The question of the ordering of operators was discussed fully in [1]. Defining χ by means of $r = \tan\chi$, $0 \leqslant \chi \leqslant \pi/2$, we are eventually led, after use of (46), to

$$H_q = -\frac{1}{2}\left[\frac{1}{\sin^5\chi\cos\chi}\frac{\partial}{\partial\chi}\sin^5\chi\cos\chi\frac{\partial}{\partial\chi} - \operatorname{cosec}^2\chi\left(4\mathcal{C}^{(2)} - \frac{1}{3}Z^2\right) - \sec^2\chi\,Z^2\right]. \tag{48}$$

The detail here is significant. The coefficient of $\mathrm{cosec}^2\chi$ has to emerge (as is seen in section 5) in an exactly suitable form, dependent only on $U(n)$ irrep labels (f, g) through their sum, for the solution process of the radial equation to work. For the irreps (f, g) of $SU(3)$, with eigenvalue $z = |f - g|$ of $Y^{(3)} = Z$, which occur in the decomposition of the $SU(4)$ irreps $(\lambda, 0, \lambda)$, according to (21) for $n = 3$, we have

$$c_{2,3}(f, g) = \tfrac{1}{3}(f^2 + fg + g^2 + 3f + 3g). \tag{49}$$

Hence, we see directly, or else by reference to (35) for $n = 3$, that the coefficient of $\mathrm{cosec}^2\chi$ in (48) is given by

$$h(h+4) \qquad h = f + g. \tag{50}$$

indeed dependent only on the sum $h = f + g$. This situation and the generalization of it for CP^n for general n enable the CP^n radial equations to be solved in terms of Jacobi polynomials in pleasing generalization of what we found in [1] for CP^2.

If we now take the CP^3 wavefunction in the form

$$\Psi(\chi, \sigma, \theta, \alpha, \beta, \gamma) = T(\chi)S(\sigma, \theta, \alpha, \beta, \gamma). \tag{51}$$

Here S denotes a spherical $SU(3)$ eigenfunction with $SU(3)$ irrep labels (f, g). This depends on α, β, γ through the factor

$$\mathrm{e}^{\mathrm{i}(\mu_1\alpha+\mu_2\beta+\mu_3\gamma)} \tag{52}$$

from which the eigenvalues of I_3, $Y^{(2)} = Y$, $Y^{(3)} = Z$ can be obtained. We find that the radial factor $T(\chi)$ satisfies the equation

$$T'' + (5\cot\chi + \tan\chi)T' - \frac{h(h+4)}{\sin^2\chi}T - \frac{z^2}{\cos^2\chi}T + 2ET = 0. \tag{53}$$

The key property of the function S that has been used, in the passage from (46) to (53), is simply the fact that it is an eigenfunction of the $SU(3)$ Casimir operator $\mathcal{C}^{(2)}$ with eigenvalue given by (49). These eigenfunctions can be expressed in terms of the polar angles, or put otherwise defined on the sphere S^5, using polar coordinates related by $K_p = rL_p$ to our variables (43), so that $\sum_{p=1}^{3} \bar{L}_p L_p = 1$. Such spherical $SU(3)$ eigenfunctions have been well discussed in [12], although we do not need the details here.

5. Results for CP^n

The pattern for general n can be directly inferred form the discussion of the case of $n = 3$ in section 4. Defining polar coordinates for CP^n in analogy with (43), and using the change $r = \tan\chi$ of variable, we are led with the aid of the key result (35) to the equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + [(2n-1)\cot\chi - \tan\chi]\frac{\mathrm{d}}{\mathrm{d}\chi} - \frac{z^2}{\cos^2\chi} - \frac{h(h+2n-2)}{\sin^2\chi}\right]T_{IY} = -2ET_{IY} \tag{54}$$

where $h = f + g$, and each pair (f, g) is associated with a class one irrep of $U(n)$ with $SU(n)$ labels $(f, 0^{n-3}, g)$ and $Z = Y^{(n)}$ eigenvalue $y^{(n)} = z = |f - g|$. The change

$$T = \sin^h\chi\cos^z\chi R \tag{55}$$

of dependent variable then gives rise to

$$\frac{\mathrm{d}^2R}{\mathrm{d}\chi^2} + [(2h+2n-1)\cot\chi - (2z+1)\tan\chi]\frac{\mathrm{d}R}{\mathrm{d}\chi} + 2WR = 0 \tag{56}$$

where

$$W = E - 2\rho(\rho+n) \qquad 2\rho = h + z = (f+g) + |f-g|. \tag{57}$$

Table 1. Data for the irrep R_1 of $SU(n+1)$.

f	g	ρ	z	$\dim(f,g)$		m	Eigenfunction
0	0	0	0	1	1	1	$P_1^{(n-1,0)}$
1	0	1	1	n	3	0	$\sin\chi\cos\chi$
0	1	1	1	n	$\bar{3}$	0	$\sin\chi\cos\chi$
1	1	1	0	n^2-1	8	0	$\sin^2\chi$

This is an equation for R whose solutions are polynomials in $\cos^2\chi$, which may be given much less conveniently in practice, in terms of $\cos 2\chi$. It is easy to solve (57) by hand obtaining energy eigenvalues as conditions for the termination of solutions as polynomials of degree m, as was done in detail in [1] for $n = 2$. However, we can compare (57) with an equation satisfied by Jacobi polynomials $P_m^{(\alpha,\beta)}$. Giving this initially in standard form [13]

$$(1-x^2)y'' + [(\beta-\alpha) - (\alpha+\beta+2)x]\,y' + \lambda_m y = 0 \tag{58}$$

where

$$\lambda_m = m(m+\alpha+\beta+1) \tag{59}$$

and $\alpha, \beta > -1$, we may change variable $x = \cos^2\chi$. This gives, for $y(\chi) = P_m^{(\alpha,\beta)}(\cos 2\chi)$,

$$y'' + [(2\alpha+1)\cot\chi - (2\beta+1)\tan\chi]\,y' + 4\lambda_m y = 0. \tag{60}$$

Comparison with (56) now provides the identifications

$$\alpha = h+n-1 \qquad \beta = z \qquad W_m = 2m(m+n+2\rho). \tag{61}$$

The subscript m has been applied to W here to emphasize its association with polynomials of degree m. It now follows from (57) and (61) that the energy eigenvalue of (54) is

$$E = 2\lambda(\lambda+n) \qquad \text{where} \quad \lambda = m+\rho. \tag{62}$$

Hence, for CP^n and given λ, we find a unique polynomial solution $P_m^{(\alpha,\beta)}$ of our radial equation of each degree m such that $0 \leqslant m \leqslant \lambda$, associated with each pair (f, g) allowed by $m = \lambda - \rho$. This exactly enumerates the irreps of $U(n)$ that occur in the decomposition of the irreps $(\lambda, 0^{n-2}, \lambda)$ of $SU(n+1)$. Thus we find that the stated degeneracy structure of the energy eigenspaces of the CP^n Schrödinger equation has materialized via separation of variables.

The values of the labels α, β of the $P_m^{(\alpha,\beta)}$ mentioned in the previous paragraph are given by

$$\alpha = f+g+n-1 \qquad z = |f-g|. \tag{63}$$

In the case of $\lambda = 1$, we have $m = 0, 1$. For $m = 1$, $\rho = 0$, so that only $(f, g) = (0, 0)$ in this case. For $m = 1$, $\rho = 1$, which allows $(f, g) = (1, 0) = (1, 1) = (0, 1)$. This allows us to observe each of the four irreps of $U(n)$ known to occur in the decomposition of the irrep $R_1 = (1, 0^{n-2}, 1) = ad$ of $SU(n+1)$.

Finally, we display tables containing some detailed information regarding the two lowest non-trivial energy levels of the CP^n Schrödinger equation, for the values $\lambda = 1, 2$ (see tables 1 and 2). Column five of the tables gives $\dim(f, g)$ for general n, and column six gives results for $n = 3$ and $CP^3 = SU(4)/U(3)$. In the latter case

$$R_1 = ad \qquad \dim ad = 15 \qquad R_2 = (2, 0, 2) \qquad \dim R_2 = 84. \tag{64}$$

Also column six gives radial eigenfunctions with a normalization in which the constant term has been set equal to one. It is easy, as in [1], to produce orthonormal eigenfunctions,

Table 2. Data for the irrep R_2 of $SU(n+1)$.

f	g	ρ	z	$\dim(f,g)$		m	Eigenfunction
0	0	0	0	1	1	2	$P_2^{(n+1,0)}$
1	0	1	1	n	3	1	$\sin\chi\cos\chi\, P_1^{(n,1)}$
2	0	2	2	$\frac{1}{2}n(n+1)$	6	0	$\sin^2\chi\cos^2\chi$
0	1	1	1	n	$\bar{3}$	1	$\sin\chi\cos\chi\, P_1^{(n,1)}$
1	1	1	0	n^2-1	8	1	$\sin^2\chi\, P_1^{(n+1,0)}$
2	1	2	1	$\frac{1}{2}n(n+2)(n-1)$	15	0	$\sin^3\chi\cos\chi$
0	2	2	2	$\frac{1}{2}n(n+1)$	$\bar{6}$	0	$\sin^2\chi\cos^2\chi$
1	2	2	1	$\frac{1}{2}n(n+2)(n-1)$	$\overline{15}$	0	$\sin^3\chi\cos\chi$
2	2	2	0	$\frac{1}{4}n^2(n-1)(n+3)$	27	0	$\sin^4\chi$

orthogonality being a consequence of the Sturm–Liouville nature of the various separation equations, e.g., (54).

In the tables, we note that the dimensions add up correctly

$$\begin{aligned} \dim(1,0,1) &= n(n+2) \\ \dim(2,0,2) &= \tfrac{1}{4}n^2(n+1)(n+3) \end{aligned} \tag{65}$$

and give the values 15 and 84 for CP^3.

Also, we have

$$\begin{aligned} P_1^{(f+g+n-1,|f-g|)} &= 1-\left(\frac{2\rho+n+1}{|f-g|+1}\right)\cos^2\chi \\ P_2^{(f+g+n-1,|f-g|)} &= 1-2\left(\frac{(2\rho+n+2)}{|f-g|+1}\right)\cos^2\chi+\left(\frac{2\rho+n+2}{|f-g|+1}\right)\left(\frac{2\rho+n+3}{|f-g|+2}\right)\cos^4\chi . \end{aligned} \tag{66}$$

Results in the tables can be completed using (66) and $2\rho = h+z = f+g+|f-g|$.

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